Kӧthe-Toeplitz And Topological $c_0^2(X,\lambda,p)$, $c^2(X,\lambda,p)$ and $l_\infty^2(X,\lambda,p)$

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***Abstract*— This paper is in continuation of [4]. Here we characterize generalized Kӧthe-Toeplitz duals of the matrix classes $c_0^2(X,\lambda,p)$, $c^2(X,\lambda,p)$ and $l_\infty^2(X,\lambda,p)$ and by application of these dualsof the matrix spaces $c_0^2(X,\lambda,p)$ and $c^2(X,\lambda,p)$.**

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***Index Terms*— Locally convex space , matrix space ,generalized Kӧthe-Toeplitz duals and topological duals.**

I. INTRODUCTION

Concerning the notations and terminology and results, we follows [1,3]. Let $(X, \mathfrak{I})$ be a Hausdorff locally convex topological vector space (lc TVS) over the field of complex numbers C and X^* be its topological dual .We denote $\mathbf{U}$ by the fundamental system fbalanced, convexanda bsorbingneighbourhoodsofzerovecto θ *to denote* g_U *to denote the gauge (Minkowski functionals)* generating the topology $\mathfrak{I}$ of X .

By a generalizd matrix , a generalized double sequence we mean a double sequence $\bar{x} = (x_{mn})$ with elements from X. Let p= (p_{mn}) be a double sequence of strictly positive real numbers and $\lambda = (\lambda mn)$ be a double sequence of non-zero complex numbers.Throughtout the paper we shall take p=$(p_{mn}) \in l_\infty^2$, space all bounded scalar double sequences , H= H(p)= $sup_{m,n}\, p_{mn}$ and M= M(p) = $\max(1,H)$. For x $\in$ X ,δ^{mn}(x) denotes the double sequence whose all terms are x,(see[4]).

We now consider the dual system (X, X^*) with respect to the canonical bilinear functional $\langle x,f\rangle$ which is the value of f $\in X^*$ at x$\in X$. If A$\subset X$ then polar of A is denoted to be By space of vector double sequences E(X) we mean a vector space of double sequences in X over C with respect to coordinatewise addition and scalar multiplication . The double summation $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}$ denote by $\sum\sum$ is taken in the sense $\lim_{n\to\infty}\sum\ \sum_{2\le m+n\le N}$.

$A^o = \{ f : |\langle x,f\rangle| \le 1\ for\ all\ x \in X \}$.

We take X^* with the strong topology $\beta(X^*,X)$ generated by the family D' = ={ g_{B^o} : B $\in$ B }where B is the collection of all bounded sets (or σ(X, X^*) -bounded sets) B of X , B^o is the polar of B with respect to bilinear form $\langle x,f\rangle$ =f(x) of the pairing (X, X^*) and for f$\in X^*$,

$$g_{B^o}(f) = \sup \{ |\langle x,f\rangle| : x \in B\}.$$

A subset A of linear functional which are defined on lcTVS X is called equicontinuous if there exists U$\in\mathbf{U}$ such that $A \subset U^o$. A locally convex topological vector space X is said to be sequentially barrelled if every sequence $\{f_{mn}\} \subset X^*$ which converges to θ in $\beta(X^*,X)$ is equicontinuous . For U$\in\mathbf{U}$,the set U^o is balanced ,bounded ,convex and $\beta(X^*,X)$ -complete subset of X^* .Let N(U)= { x $\in$ X : g_U (x) =0 }. For p= (p_{mn}) and $\lambda = (\lambda_{mn})$ in [4] we have introduced and studied the following classes :

(1.1) $c_0^2(X,\lambda,p) = \{ \bar{x} = (x_{mn}) : x_{mn} \in X , m,n \ge 1$ and $(g_U(\lambda_{mn} x_{mn}))^{p_{mn}} \to 0$ as m +n $\to\infty$ for each $g_U \in D$ };

(1.2) $c^2(X,\lambda,p) = \{ \bar{x} = (x_{mn}) : x_{mn} \in X , m,n \ge 1$ and $(g_U(x_{mn}\lambda_{mn} - x))^{p_{mn}} \to 0$ m+ n $\to\infty$ for each $g_U \in D$ };

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(1.3) $l_\infty^2(X,\lambda,p) = \{ \bar{x} = (x_{mn}) : x_{mn} \in X, m,n \geq 1$ and

$$\sup_{m,n} (g_U(x_{mn}\lambda_{mn}))^{p_{mn}} < \infty \text{ for each } g_U \in D \}.$$

Then the quotient spaces X_U = X/ N(U) is a normed space with respect to the norm $\hat{g}$ where $\hat{g}$ x(U))= g_U (x) , x(U)being the equivalence class in X_U corresponding to the element x $\in X$. The subspace $X^*(U^o) = \bigcup_{n=1}^{\infty} nU^o$ of X^*, is a Banach space with respect to the norm g_{U^o} (f) = sup{ $|\langle x,f\rangle|$: x $\in U$}. Further we have

THEOREM 1.1: The Banach space ($X^*(U^o)$, g_{U^o}) is the topological dual of (X_U , $\hat{g}_U$) for each U $\in$ U.

We now define the generalized KÖthe-Toeplitz duals i.e., generalized $\alpha-, \beta-, and\ \gamma-duals$ for a class E(X) of vector double sequences by

$(E(X))^\alpha = \{ \bar{f} = (f_{mn}) : f_{mn} \in X^*$, m ,n ≥ 1 and $\sum\sum |\langle x_{mn}, f_{mn}\rangle| < \infty$ for all $\bar{x} = (x_{mn}) \in$ E(X) };

$(E(X))^\beta = \{ \bar{f} = (f_{mn}) : f_{mn} \in X^*$, m ,n ≥ 1 and $\sum\sum \langle x_{mn}, f_{mn}\rangle$ *is convergent* for all $\bar{x} = (x_{mn}) \in$ E(X) };

$(E(X))^\gamma = \{ \bar{f} = (f_{mn}) : f_{mn} \in X^*$, m ,n ≥ 1 and $\sup_{N>1} |\sum\sum_{2\leq m+n\leq N} \langle x_{mn}, f_{mn}\rangle|$ for all $\bar{x} = (x_{mn}) \in$ E(X) }.

DEFINITION 1.2 : Let E(X) be a space of vector double sequences .(i) E(x) is said to be normal if for $\bar{x} = (x_{mn}) \in$ E(X) and for every scalar double sequence $\bar{\alpha} = (\alpha_{mn})$ with $|\alpha_{mn}| \leq 1$, m,n ≥ 1 the double sequence $\bar{\alpha}\bar{x} = (\alpha_{mn}x_{mn}) \in$ E(X). (ii) E(X) if is said to be monotone E(X) contains the canonical pre – images of all its step spaces (cf.[2]).

On the lines of scalar single sequences [2], we can easily prove :

THEOREM 1.3 : A space E(x) of vector double sequences is

(i) normal if and only if l_∞^2 E(X) $\subset$ E(X) ;and

(ii) monotone if and only if m_0^2 E(X) $\subset$ E(X) ,

where m_0^2 is the space of scalar double sequences spanned by all double sequences formed by zeros and ones.

Further we easily get :

THEOREM 1.4 : (i) $(E(X))^\alpha \subset (E(X))^\beta \subset (E(X))^\gamma$,

(ii) $(E(X))^\alpha = (E(X))^\beta$ if E(X) is monotone , and

(iii) $(E(X))^\alpha = (E(X))^\gamma$ if E(X) is normal .

II. $K\ddot{O}THE - TOEPLITZ$ DUALS

In this section we characterize $\alpha-, \beta-, and\ \gamma-duals$ $c_0^2(X,\lambda,p)$, $c^2(X,\lambda,p)$ and $l_\infty^2(X,\lambda,p)$.

We easily have :

LEMMA 2.1 : (I) $c_0^2(X,\lambda,p)$ and $l_\infty^2(X,\lambda,p)$ are normal ; and

(ii) $c^2(X,\lambda,p)$ is not monotone

We now define

(2.1) $M_0^2(X,\lambda,p) = \{ \bar{f} = (f_{mn}) : f_{mn} \in X^*$, m , n ≥ 1 and for each B $\in$ B there exists an integer K > 1 such that $\sum\sum |\lambda|^{-1} g_{B^o}(f_{mn}) K^{-1/p_{mn}} < \infty$ }.

THEOREM 2.2 : If X sequentially barrelled lcTVS then $(c_0^2(X,\lambda,p))^\alpha = M_0^2(X^*,\lambda,p)$.

COROLLARY 2.3 : If X is sequentially barrelled lcTVS then

$$(c_0^2(X,\lambda,p)^\beta = (c_0^2(X,\lambda,p)^\gamma = M_0^2(X,^*\lambda,p).$$

THEOREM 2.4 : Let X be sequentially barrelled lcTVS.Then

(i) $(c_0^2(X,\lambda,p))^\alpha = M_0^2(X^*,\lambda,p) \cap S(X^*,\lambda, l_1^2)$

(ii) $(c_0^2(X,\lambda,p))^\beta = M_0^2(X^*,\lambda,p) \cap S(X^*,\lambda,(cs)^2)$

(iii $(c_0^2(X,\lambda,p))^\gamma = M_0^2(X^*,\lambda,p) \cap S(X^*,\lambda,(bs)^2)$.

COROLLARY 2.5 : If inf $p_{mn} > 0$ and X is sequentially barrelled lcTVS then

$(c_0^2(X,\lambda,p))^\alpha = (c_0^2(X,\lambda,p)^\beta = (c_0^2(X,\lambda,p)^\gamma = l_1^2(X^*,\lambda)$.

$where \quad l_1^2(X^*,\lambda) = \{\bar{f} = (f_{mn}) : f_{mn} \in X^*$, m, n, $\sum\sum |\lambda_{mn}|^{-1} g_{B^o}(f_{mn}) < \infty$ for each B $\in$ B } .

For the next theorem we define

(2.2) $M_\infty^2(X^*, \lambda, p) = \{ \bar{f} = (f_{mn}) \in X^*$, m, n ≥ 1 such that for each B $\in$ B and for each K > 1, $\sum\sum|\lambda_{mn}|^{-1} g_{B^o}(f_{mn}) K^{-1/p_{mn}} < \infty \}$

THEOREM 2.6 : If X is sequentially barreled lc TVS then

$$(l_\infty^2(X, \lambda, p))^\alpha = M_\infty^2(X^*, \lambda, p).$$

Moreover from Lemma 2.1 and Theorems 1.4 and 2.6 ,we easily get :

COROLLARY 2.7 : If X is sequentially barreled lcTVS then

$$(l_\infty^2(X, \lambda, p))^\beta = (l_\infty^2(X, \lambda, p))^\gamma = M_\infty^2(X^*, \lambda, p).$$

III. CONTINUOUS DUAL

In the following Theorems continuous duals of $c_0^2(X, \lambda, p)$ and $c^2(X, \lambda, p)$ are characterized by applications of the results concerning $Köthe - Toeplitz$ duals obtained in section 2.

THEOREM 3.1 : If X is sequentially barreled lcTVS then the topological dual ($c_0^2(X, \lambda, p))^*$ of ($c_0^2(X, \lambda, p)$, $\sigma g)$ is isomorphic to $M_0^2(X^*, \lambda, p)$.

THEOREM 3.2 : If inf p_{mn} >0 and X is sequentially barreled lcTVS then F $\in c^2(X, \lambda, p))^*$. ,the topological dual of $(c^2(X, \lambda, p), \sigma g)$, if and only if there exists f $\in X^*$ and $\bar{f} = (f_{mn}) \in l_1^2(X^*,\lambda)$ such that for each $\bar{x} = (x_{mn}) \in c^2(X, \lambda, p)$

$$F(\bar{x}) = \langle x, f\rangle + \sum\sum\langle x_{mn}, f_{mn}\rangle$$

where x $\in$ X satisfies $(g_U(x_{mn}\lambda_{mn} - x))^{p_{mn}} \to 0$ as m +n $\to \infty$ $for\ each\ g_U \in$ D.

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